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Expanded uncertainty evaluation taking into account the correlation between estimates of input quantities

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Abstract

An expression for estimating the combined standard uncertainty taking into account the observed correlation between the estimates of the two input quantities is given. The Welch – Satterthwaite formula given in the GUM is analyzed. It is shown that the number of degrees of freedom calculated using this formula will vary over a wide range when the value of the correlation coefficient changes, and in some cases it may take an unacceptable zero value. An expression for calculating the combined standard uncertainty by the reduction method is given. It is shown that the number of degrees of freedom in this method does not depend on the value of the correlation coefficient.

A formula for calculating the effective number of degrees of freedom taking into account the observed correlation is proposed. The existing expression for calculating the kurtosis of the measurand is analyzed and an expression is proposed for calculating the kurtosis of the measurand in the presence of a correlation between the input quantities. An example of estimation of expanded uncertainty when measuring the coefficient of a pressure transducer using a calibrator is considered. Estimates of the distribution of the measurand, obtained using Monte Carlo simulation, showed that they are closest to the estimates obtained by the kurtosis method. The considered example showed that taking into account the correlation in the processing of measurement results makes it possible to reduce the expanded measurement uncertainty of the converter coefficient by 1.22–1.27 times.

Keywords: measurement uncertainty; correlation; effective number of degrees of freedom; method of kurtosis.

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1. Introduction

When evaluating the measurement uncertainty, one has to deal with situations where estimates of input quantities are pairwise correlated. When calculating the standard uncertainty $u(y)$ of the measurand $Y = f(X_1, X_2, ..., X_N)$, the correlation between the estimates of l-th and k-th input quantities is taken into account using the well-known formula [1]:

$$u(y) = \sqrt{\sum_{j=1}^{N} c_j^2 u_j^2 + 2r_{l,k} c_l c_k u_l u_k}, \qquad (1)$$

where u_j, c_j are the standard uncertainties and sensitivity coefficient of j-th input quantities, $j = 1, 2, ..., N$; $r_{l,k}$ is correlation coefficient. Difficulties in accounting for correlation arise when evaluating expanded uncertainty U. The latter for linearized models is defined as the product of the standard uncertainty $u(y)$ by the coverage factor k:

$$U = ku_c(y). \qquad (2)$$

The coverage factor k is determined in different ways with different approaches to estimating measurement uncertainty.

2. GUM approach

In GUM [1], the Student's coefficient $t_p(\nu_{eff})$ for the given confidence level p and the effective number of degrees of freedom ν_{eff} is taken as the coverage coefficient k for repeated measurements. The ν_{eff} is obtained by the Welch–Satterthwaite formula:

$$\nu_{eff} = \frac{u^4(y)}{\sum_{j=1}^{N} \frac{c_j^4 u_j^4}{\nu_j}}, \qquad (3)$$

where ν_j is the number of degrees of freedom of the j-th input quantity. Expression (3) does not give a correct estimate of the number of degrees of freedom in the presence of a correlation between the input quantities. Indeed, for a function of two correlated

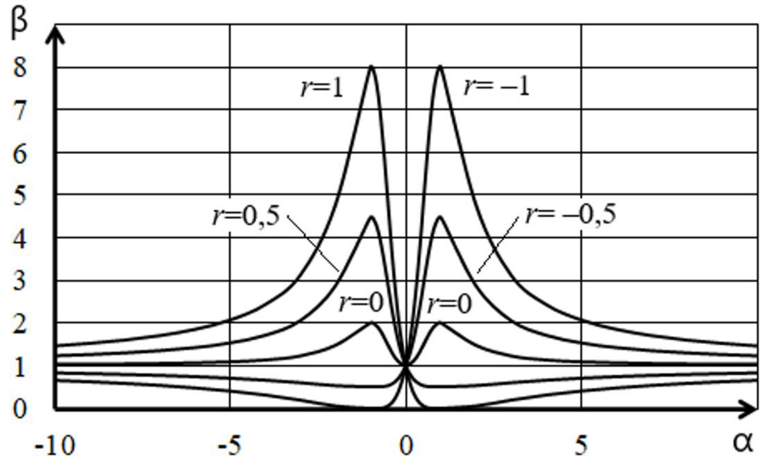

Fig. 1. Dependence of the coefficient β in expression (5) on α, when changing the correlation coefficient in the range [-1;1]

input quantities $Y = f(X_1, X_2)$ with an equal number of degrees of freedom $\nu_1=\nu_2=\nu$ and in the absence of uncertainties of type B, the effective number of degrees of freedom will be equal to

$$\nu_{eff} = \nu \frac{(c_1^2u_1^2 + 2r_{1,2}c_1c_2u_1u_2 + c_2^2u_2^2)^2}{c_1^4u_1^4 + c_2^4u_2^4}. \quad (4)$$

Denoting the ratio $\frac{c_1u_1}{c_2u_2} = \alpha$, we can rewrite (4) as:

$$\nu_{eff} = \nu \frac{(\alpha^2 + 2r_{1,2}\alpha + 1)^2}{\alpha^4 + 1} = \nu\beta. \quad (5)$$

The dependences of the coefficient β in expression (5) on α when the correlation coefficient changes in the range $-1 \le r \le 1$ are shown in Fig. 1.

In Fig. 1 it can be seen that the effective number of degrees of freedom ν_{eff} will change in the range from 0 to 8 ν when changed $-1 \le r \le 1$. It should be noted that the value of the Student's coefficient for the number of degrees of freedom equal to zero does not make sense at all.

On the other hand, to calculate the combined standard uncertainty of the presence of correlation, the reduction method can be used [2]. It provides for bringing indirect measurements to direct ones by calculating the values of the measured value for each pair of correlated input quantities:

$$y_i = f(x_{1i}, x_{2i}), \quad i = 1, 2, ..., n. \quad (6)$$

In this case, the measured value will be the arithmetic mean of the measured values obtained:

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i, \quad (7)$$

and the standard uncertainty of type A of the measured quantity is found as:

$$u_A(y) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(y_i - \bar{y})^2}, \quad (8)$$

and has the number of degrees of freedom ν = n–1, which should be equal to the number of degrees of freedom ν_{eff}, determined by the Welch – Satterthwaite formula (4). This situation can be tuned when taking into account that the correlated input quantities must be described by the joint PDF, which contributes $u_{l,k}(y)$ to the standard uncertainty of the measurand with the number of degrees of freedom $\nu = n-1$. In this case, the expression for the combined standard uncertainty (1) can be rewritten as follows:

$$u(y) = \sqrt{\sum_{\substack{j=1,\\ j\neq k\neq l}}^{N} c_j^2u_j^2 + [c_l^2u_l^2 + 2r_{l,k}c_lc_ju_lu_k + c_k^2u_k^2]} = \sqrt{\sum_{\substack{j=1,\\ j\neq k\neq l}}^{N} c_j^2u_j^2 + u_{l,k}^2(y)}. \quad (9)$$

In this case, the Welch – Satterthwaite formula in the presence of correlated input quantities will have the form:

$$\nu_{eff} = \frac{u^4(y)}{\sum_{\substack{j=1,\\ j\neq k\neq l}}^{N} \frac{c_j^4u_j^4}{\nu_j} + \frac{u_{l,k}^4}{n-1}}. \quad (10)$$

So, for a function of two correlated input quantities with an equal number of degrees of freedom $\nu_1=\nu_2=\nu$, the effective number of degrees of freedom will be equal to ν, which coincides with the number of degrees of freedom for expression (8).

3. Kurtosis method

The most reliable estimate of expanded uncertainty can be obtained on the basis of the law of propagation of distributions by the Monte Carlo method (MCM) [3–4]. However, its implementation by existing software [5] does not allow obtaining an uncertainty budget with estimates of the contributions of the input quantities uncertainties to the measurand uncertainty. The kurtosis method allows obtaining estimates of expanded uncertainty compatible with estimates of MCM

Table 1

Results of simultaneous measurement of pressure and current

P, MPa	0.10001	0.10001	0.10000	0.10000	0.10002
I, mA	8.008	8.008	8.006	8.006	8.010
P, MPa	0.10000	0.10002	0.10000	0.10000	0.10000
I, mA	8.006	8.01	8.006	8.006	8.006

[6]. In this method, the expanded uncertainty is found by the formula:

$$U = k(\eta)\cdot u(y), \tag{11}$$

where the coverage factor $k(\eta)$ depends on the kurtosis η of the measurand, determined by the formula:

$$\eta = \frac{1}{u^4(y)}\sum_{j=1}^{N} c_j^4 u_j^4 \eta_j, \tag{12}$$

where η_j is kurtosis of the j-th input quantity. This expression also does not work in the case of correlated input quantities; however, it can be transformed for this case by analogy with expression (10):

$$\eta = \frac{\sum\limits_{\substack{j=1,\\ j\neq k,\, j\neq l}}^{N} c_j^4 u_j^4 \eta_j + \eta_{l,k} u_{l,k}^2 (y)^2}{u^4(y)}. \tag{13}$$

The coverage factor for a confidence level of 0.95 is calculated by the formula [5]:

$$k = \begin{cases} 0.1085\eta^3 + 0.1\eta + 1.96, \text{ at } \eta < 0; \\ t_{0.95;(6/\eta+4)} \cdot \sqrt{\dfrac{3+\eta}{3+2\eta}}, \quad \text{at } \eta \geq 0. \end{cases} \tag{14}$$

4. Example. Determination of the transducer factor of the PC-28 pressure sensor

To determine the transducer factor γ of the pressure sensor, a DPI-802 pressure calibrator is used. The results of multiple simultaneous measurements of pressure P and current I are given in Table 1.

The correlation coefficient between the pressure and current results of measuring is 1. The uncertainty budgets for the measuring the transducer factor $\gamma = I/P$ in accordance with the GUM and kurtosis method are given in Tables 2, 3. The estimates of the standard uncertainties of the input quantities by type B were obtained from the maximum permissible error of the input quantities under the assumption of their uniform PDF.

The value of the expanded uncertainty excluding correlation is U=0.019 mA/MPa.

Table 2

The uncertainty budget for the measurement uncertainty evaluation of the transducer factor according to the GUM method

Input quantity	Estimation of input quantity	Standard uncertainty	Number of degrees of freedom	Sensitivity coefficient	Uncertainty contribution, mA/MPa
I, mA	8.0072	0.000533	9	9.9994	0.0053
		0.0005774	∞		0.0050
P, MPa	0.100006	0.00000267	9	-800.6239	-0.0021
		0.000005774	∞		-0.0052
Measurand	Estimation of measurand	Combined standard uncertainty	Effective number of degrees of freedom	Coverage factor	Expanded uncertainty, mA/MPa
γ, mA/MPa	80.067	0.008057	363	1.97	0.0159

Table 3

The uncertainty budget for measurement uncertainty evaluation of the transducer factor by the kurtosis method

Input quantity	Estimation of input quantity	Standard uncertainty	Kurtosis of input quantity	Sensitivity coefficient	Uncertainty contribution, mA/MPa
I, mA	8.0072	0.000605	1.2	9.9994	0.00605
		0.0005774	-1.2		0.00577
P, MPa	0.100006	0.00000302	1.2	-800.6239	-0.00242
		0.000005774	-1.2		-0.00462
Measurand	Estimation of measurand	Combined standard uncertainty	Kurtosis of measurand	Coverage factor	Expanded uncertainty, mA/MPa
γ, mA/MPa	80.0672	0.008237	-0.364	1.92	0.0158

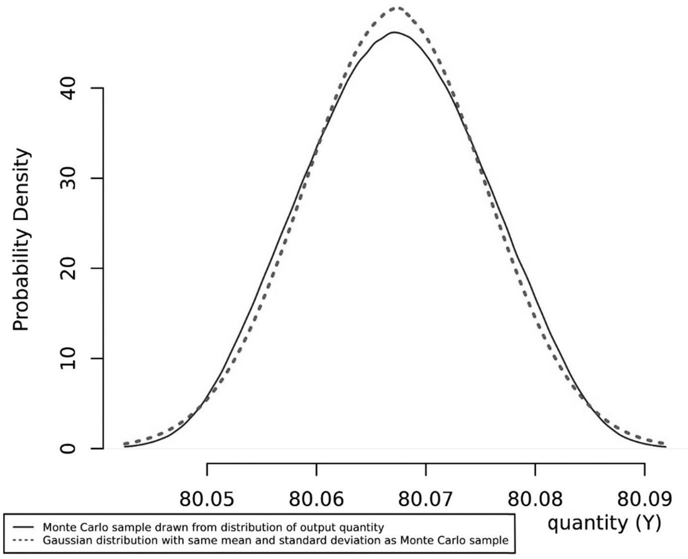

Fig. 2. Result of Monte-Carlo simulation

Monte Carlo simulation [5] (Fig. 2) gives us the following estimates of the distribution of the measured quantity: MMK $\gamma = 80.0672$; $U = 0.0157$; $u(y) = 0.00824$; $k = 1.91$. These estimates are the closest to those obtained by the kurtosis method.

The value of the expanded uncertainty excluding correlation is $U = 0.023$ mA/MPa. Thus, taking correlation into account when processing the measurement results allows us to reduce the expanded measurement uncertainty of the transducer factor by 1.22–1.27 times.

Conclusions

1. Expanded uncertainty evaluation according to the GUM method does not allow obtaining a reliable estimate of the expanded uncertainty of measurements in the presence of an observed correlation between the values of the input quantities.

2. Analysis of the joint probability density function of the function of correlated input quantities and the reduction method allowed us to propose a formula for calculating the effective number of degrees of freedom, taking into account the observed correlation between the values of the input quantities.

3. The proposed approach made it possible to obtain a formula for estimating the expanded uncertainty by the kurtosis method, taking into account the observed correlation between the values of the input quantities.

4. An example of estimating the expanded uncertainty in determining the coefficient of the PC-28 pressure sensor transducer is considered. Estimates of the distribution of the measured quantity obtained using the Monte Carlo simulation showed that they are closest to the estimates obtained by the kurtosis method.

5. The considered example has shown that taking into account the correlation in the processing of measurement results makes it possible to reduce the expanded measurement uncertainty of the converter coefficient by a factor of 1.22–1.27.

Оцінювання розширеної невизначеності вимірювань з урахуванням кореляції між оцінками вхідних величин

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Анотація

Наведено вираз для оцінювання сумарної стандартної невизначеності з урахуванням спостережуваної кореляції між оцінками двох вхідних величин. Проаналізовано наведену в GUM формулу Велча-Саттерсвейта. Показано, що розраховане за цією формулою число ступенів свободи буде змінюватися в широких межах при зміні значення коефіцієнта кореляції, а в деяких випадках може приймати неприпустиме нульове значення. Наведено вираз для обчислення сумарної стандартної невизначеності методом редукції. Показано, що число ступенів свободи в цьому методі не залежить від значення коефіцієнта кореляції. Таким чином, оцінювання розширеної невизначеності за методикою GUM не дозволяє отримати достовірну оцінку розширеної невизначеності вимірювань за наявності кореляції, що спостерігається між значеннями вхідних величин. Запропоновано формулу для розрахунку ефективного числа ступенів свободи з урахуванням спостережуваної кореляції. Проаналізовано існуючий вираз для обчислення ексцесу вимірюваної величини і запропоновано вираз для розрахунку ексцесу вимірюваної величини за наявності кореляції між вхідними величинами. Розглянуто приклад оцінювання розширеної невизначеності при вимірюванні коефіцієнта перетворювача давача тиску за допомогою калібратора. Оцінки розподілу вимірюваної величини, отримані за допомогою моделювання методом Монте-Карло, показали, що вони найбільш близькі до оцінок, отриманих методом ексцесу. Розглянутий приклад показав, що облік кореляції при обробці результатів вимірювань дозволяє знизити розширену невизначеність вимірювання коефіцієнта перетворювача в 1,22–1,27 рази.

Ключові слова: невизначеність вимірювань; кореляція; ефективне число ступенів свободи; метод ексцесів.

Оценивание расширенной неопределенности измерений с учетом корреляции между оценками входных величин

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Аннотация

Приведено выражение для оценивания суммарной стандартной неопределенности с учетом наблюдаемой корреляции между оценками двух входных величин. Проанализирована приведенная в GUM формула Велча-Саттерсвейта.

Предложена формула для расчета эффективного числа степеней свободы с учетом наблюдаемой корреляции. Проанализировано существующее выражение для вычисления эксцесса измеряемой величины и предложено выражение для расчета эксцесса измеряемой величины при наличии корреляции между входными величинами. Рассмотрен пример оценивания расширенной неопределенности при измерении коэффициента преобразователя датчика давления с помощью калибратора. Оценки распределения измеряемой величины, полученные с помощью моделирования методом Монте-Карло, показали, что они наиболее близки к оценкам, полученным методом эксцесса. Рассмотренный пример показал, что учет корреляции при обработке результатов измерений позволяет снизить расширенную неопределенность измерения коэффициента преобразователя в 1,22–1,27 раза.

Ключевые слова: неопределенность измерения; корреляция; эффективное число степеней свободы; метод эксцесса.

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