

Analysis of approaches to calculating effective degrees of freedom in the presence of correlation between the results of measuring input quantities

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Abstract

The paper analyses available approaches to determining effective degrees of freedom when evaluating the expanded uncertainty of correlated measurements: the GUM approach, the expressions proposed in the papers by R. Willink and H. Kastrup, the approach proposed by the Authors, and the reduction method.

The simplest mathematical model with two input quantities was chosen as a test model, the results of which from n simultaneous observations, distributed according to the normal law, are assigned only to Type A uncertainties. For this model, the measurand corresponds to a bivariate t -distribution with $(n-1)$ degrees of freedom.

The analysis suggested that the Welch-Satterthwaite equation presented in the GUM approach, as well as the expressions proposed in the papers by R. Willink and H. Kastrup, lead to an effective degree of freedom that differs from $(n-1)$ when the correlation coefficient and the ratio of the input quantity uncertainty contributions vary, demonstrating the unreliability of these expressions. It was shown that the number of degrees of freedom $(n-1)$ is obtained using the Authors' proposed method and the reduction method developed specifically for correlated measurements.

The paper presents an example of an extended uncertainty evaluation for resistor resistance measurements using the voltmeter-ammeter method, demonstrating the agreement between the results obtained using the method proposed by the Authors and the reduction method.

Keywords: measurement uncertainty; correlation; effective degrees of freedom; Welch-Satterthwaite equation; reduction method.

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1. Introduction

One of the main principles for evaluating the expanded uncertainty in the Guide to the Expression of Uncertainty in Measurement (GUM) [1], with the presence of significant Type A uncertainty contributions, is to assign a t -distribution to the measured quantity with degrees of freedom ν_{eff} determined by the Welch-Satterthwaite (W-S) equation.

In [2, 3], it was shown that for correlated data, under certain realistic conditions, this formula leads to a value equal to zero, which corresponds to an impossible value of expanded uncertainty equal to infinity.

The expressions for calculating ν_{eff} in the presence of an observed correlation between the measurement results of input quantities, published in [4, 5], are analysed.

For the analysis, the following simple test mathematical model is used

$$Y = f(X_1, X_2) \quad (1)$$

of a linearizable function of two input quantities measured simultaneously n times, the measurement results of which are correlated with the correlation coefficient $r_{1,2}$ and include only type A uncertainties with the same degrees of freedom $\nu = n-1$.

Result 1 of [4] applies to this case: "If any group of input quantities is estimated from a set of n repeated observations then this group of quantities shall be combined to obtain a single component of uncertainty with $n-1$ degrees of freedom".

In other words, if there is a group of L input quantities X_1, X_2, \dots, X_L the estimates (x_1, x_2, \dots, x_L) of which are obtained from n simultaneously performed measurements x_{iq} ($i = 1, 2, \dots, L; q = 1, 2, \dots, n$), distributed according to the normal law, then this group shall be described by a joint L -dimensional t -distribution law with the degrees of freedom $\nu = n-1$ [6].

Result 1 [4], as shown in Section 6 of this paper, is confirmed by the implementation of the reduction method described in [7].

Below, the analysis of various approaches to estimating the expanded uncertainty of correlated measurements based on the test model (1) is considered.

2. The GUM approach

A real, explicit, one-dimensional, linear, or linearizable mathematical measurement model is as follows:

$$Y = f(X_1, X_2, \dots, X_N), \quad (2)$$

where Y is the measurand; X_1, X_2, \dots, X_N are the input quantities.

Having estimates of the input quantities x_1, x_2, \dots, x_N , one can substitute them into equation (1) to obtain an estimate of the measurand:

$$y = f(x_1, x_2, \dots, x_N). \quad (3)$$

Based on the distribution law for variances and covariances, the standard uncertainty of the measurand is written as:

$$u(y) = \sqrt{\sum_{i=1}^N u_{Ai}^2(y) + 2r_{l,m} u_{Al}(y) u_{Am}(y) + \sum_{i=1}^N u_{Bi}^2(y)}, \quad (4)$$

where $u_{Ai}(y) = c_i u_A(x_i)$, $u_{Bi}(y) = c_i u_B(x_i)$ are the contributions of Type A $u_A(x_i)$ and Type B $u_B(x_i)$ measurement uncertainties to the uncertainty of the measurand $u(y)$; $r_{l,m}$ is the observed correlation coefficient between the results of simultaneous measurements of the l -th and m -th input quantities; c_i is the corresponding sensitivity coefficient.

Expanded uncertainty is determined as:

$$U(y) = t_{p;v_{eff}} u(y), \quad (5)$$

where $t_{p;v_{eff}}$ is the t -coefficient for the confidence level p and the effective degrees of freedom v_{eff} , calculated according to the W-S equation:

$$v_{eff} = \frac{u^4(y)}{\sum_{i=1}^N \frac{u_{Ai}^4(y)}{n_i - 1} + \sum_{i=1}^N \frac{u_{Bi}^4(y)}{v_{Bi}}}, \quad (6)$$

in which $v_{Ai} = n_i - 1$, v_{Bi} are the degrees of freedom assigned, respectively, to the standard uncertainties of Type A and B of the i -th input quantity.

Applying test model equation (1) to expression (6) yields the following result:

$$v_{eff} = v \frac{(c_1^2 u_1^2 + 2rc_1 c_2 u_1 u_2 + c_2^2 u_2^2)^2}{c_1^4 u_1^4 + c_2^4 u_2^4}. \quad (7)$$

Denoting the ratio $\frac{c_1 u_1}{c_2 u_2} = \alpha$, one can rewrite (7) as:

$$v_{eff} = v \frac{(\alpha^2 + 2r_{1,2}\alpha + 1)^2}{\alpha^4 + 1} = v\beta. \quad (8)$$

The dependences of the coefficient β in expression (8) on α , as the correlation coefficient varies in the range $-1 \leq r \leq 1$, are presented in [2].

The analysis of these dependences suggests that the effective degrees of freedom v_{eff} instead of the value $n-1$ declared in Result 1 [4], will vary in the range from 0 to $8v$ as $-1 \leq r \leq 1$. In this case, at two points: $r = 1$ and $\alpha = -1$, and also at a $r = -1$ and $\alpha = 1$, the value β (and, consequently, v_{eff}) will be equal to zero, which leads to the value of t -coefficient equal to infinity, which has no meaning at all.

3. R. Willink's approach

The paper [4] presents a general formula (26) for the effective degrees of freedom for a model with N pairwise correlated input quantities with correlation coefficients r_{ij} and degrees of freedom v_{ij} :

$$v_{eff} = \frac{u_y^4}{\sum_{i=1}^N \left(\sum_{j=1}^N c_i c_j r_{ij} u_i u_j \right)^2 / v_i}, \quad (9)$$

in which u_i , u_j and c_i , c_j are their standard uncertainties and the corresponding sensitivity coefficients.

Applying the test model equation (1) to expression (9) yields the following result:

$$v_{eff} = \frac{(n-1)(c_1^2 u_{A1}^2 + 2r_{12} c_1 c_2 u_{A1} u_{A2} + c_2^2 u_{A2}^2)^2}{(c_1^2 u_{A1}^2 + r_{12} c_1 c_2 u_{A1} u_{A2})^2 + (r_{12} c_1 c_2 u_{A1} u_{A2} + c_2^2 u_{A2}^2)^2}. \quad (10)$$

Denoting the ratio $\frac{c_1 u_{A1}}{c_2 u_{A2}} = \alpha$, one can rewrite (10) as:

$$v_{eff} = (n-1) \frac{(\alpha^2 + 2r_{1,2}\alpha + 1)^2}{(\alpha^2 + r_{1,2}\alpha)^2 + (1 + r_{1,2}\alpha)^2} = (n-1)\beta, \quad (11)$$

where

$$\beta = \frac{(\alpha^2 + 2r_{1,2}\alpha + 1)^2}{\alpha^2 (\alpha + r_{1,2})^2 + (1 + r_{1,2}\alpha)^2}. \quad (12)$$

The dependences of the coefficient β in expression (12) on α as the correlation coefficient varies over the range $-1 \leq r \leq 1$ are shown in Fig. 1.

Analysis of Fig. 1 shows that the expression obtained by R. Willink leads to zero values of the effective degrees of freedom at points $r = 1$, $\alpha = -1$ and $r = -1$; $\alpha = 1$ and also shows their variation within a range from 0 to $2v$ with a change in the value of the correlation coefficient and the ratio of the contributions of the uncertainties of the input quantities. This result contradicts Result 1 of R. Willink's paper [4], therefore raising doubts about the use of formula (26) in this paper to calculate the effective number of degrees of freedom for correlated measurements.

4. H. Castrup's approach

H. Castrup's paper [5] also provides a general formula (46) for v_{eff} similar to (9), in the form:

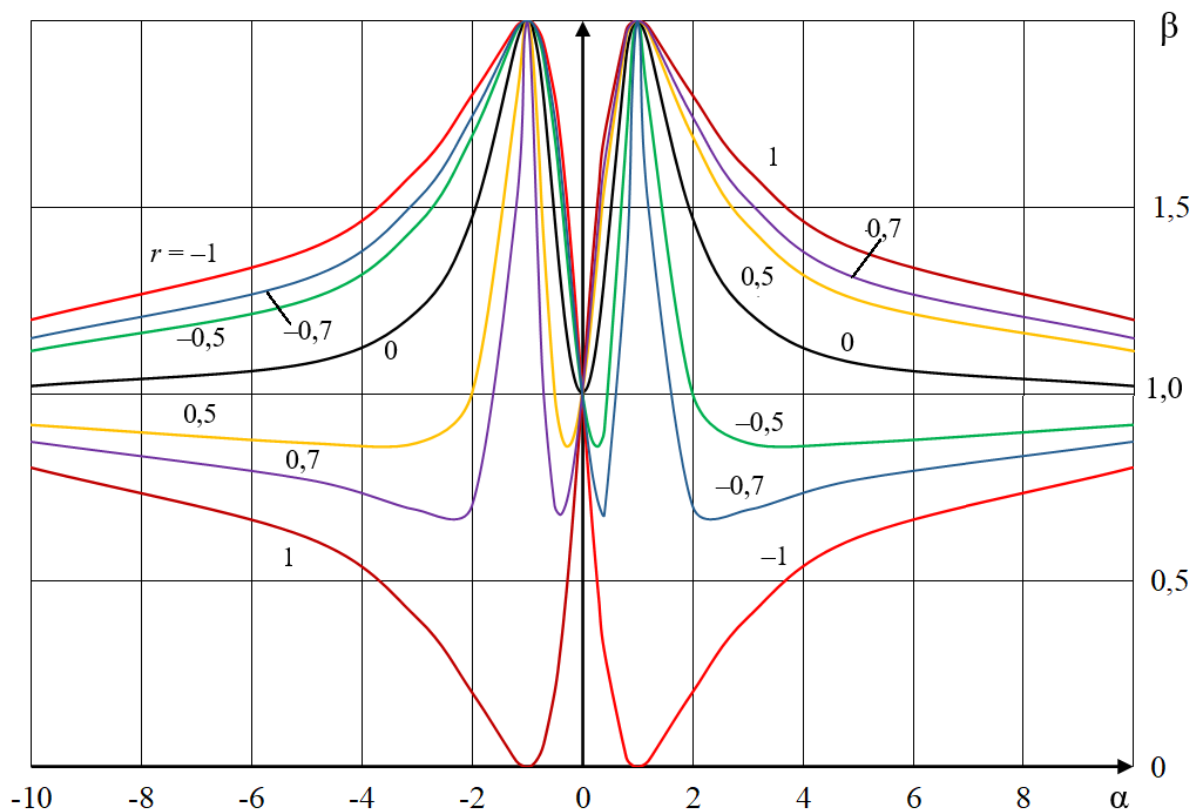


Fig. 1. Dependences of the coefficient β in expression (12) on α when changing the correlation coefficient in the range $-1 \leq r \leq 1$

$$v_{eff} \cong \frac{u_y^4}{\sum_{i=1}^N \frac{c_i^4 u_i^4}{v_i} + \sum_{i=1}^{N-1} \sum_{j>i}^N r_{ij}^2 c_i^2 c_j^2 \left(\frac{u_i^2}{v_i}\right) \left(\frac{u_j^2}{v_j}\right) \left(v_i + v_j + \frac{1}{2}\right) + 2 \sum_{i=1}^{N-1} \sum_{j>i}^N r_{ij} c_i c_j u_i u_j \left(\frac{c_i^2 u_i^2}{v_i} + \frac{c_j^2 u_j^2}{v_j}\right)}. \quad (13)$$

Applying expression (1) to equation (13) yields the following result:

$$v_{eff} \cong \frac{(n-1)(c_1^2 u_{A1}^2 + 2r_{12} c_1 c_2 u_{A1} u_{A2} + c_2^2 u_{A2}^2)^2}{c_1^4 u_1^4 + c_2^4 u_2^4 + r_{12}^2 c_1^2 c_2^2 u_1^2 u_2^2 \frac{2n-1,5}{(n-1)} + 2r_{12} c_1 c_2 u_1 u_2 (c_1^2 u_1^2 + c_2^2 u_2^2)}. \quad (14)$$

Denoting the ratio $\frac{c_1 u_{A1}}{c_2 u_{A2}} = \alpha$, one can rewrite (14) as:

$$v_{eff} \cong (n-1) \frac{(\alpha^2 + 2r_{12} \alpha + 1)^2}{\alpha^4 + 1 + r_{12}^2 \alpha^2 \frac{2n-1,5}{(n-1)} + 2r_{12} \alpha (\alpha^2 + 1)} = (n-1) \beta, \quad (15)$$

where

$$\beta = \frac{(\alpha^2 + 2r_{12} \alpha + 1)^2}{\alpha^4 + 1 + r_{12}^2 \alpha^2 \frac{2n-1,5}{(n-1)} + 2r_{12} \alpha (\alpha^2 + 1)}. \quad (16)$$

The dependences of the coefficient β in expression (16) on α as the correlation coefficient varies over the range $-1 \leq r \leq 1$ are shown in Fig. 2 for $n = 5$ and in Fig. 3 for $n = 10$.

The analysis of Figs. 2 and 3 shows that for the adopted test model, the number of effective degrees of freedom is not equal to the value $n-1$ and differs from it within wide limits with changes not only in the correlation coefficient and the ratio between the contributions of the uncertainty of the input quantities, but also with changes in the number of repeated observations n . In addition, at two points of dependence (16), for $r = 1$ and $\alpha = -1$, as well as for $r = -1$ and $\alpha = 1$, the β value (and, consequently, v_{eff}) takes on an unacceptable zero value. This result contradicts Result 1 of R. Willink's paper [4], raising doubts about the use of formula (46) in paper [5] for calculating the effective degrees of freedom for correlated measurements.

5. The Authors' approach

In [2], the Authors propose an approach to calculating the effective degrees of freedom for correlated measurements, accounting for the fact that the correlated input quantities shall be described by a joint probability density function that contributes $u_{l,m}(y)$ to the standard uncertainty of the measurand with degrees

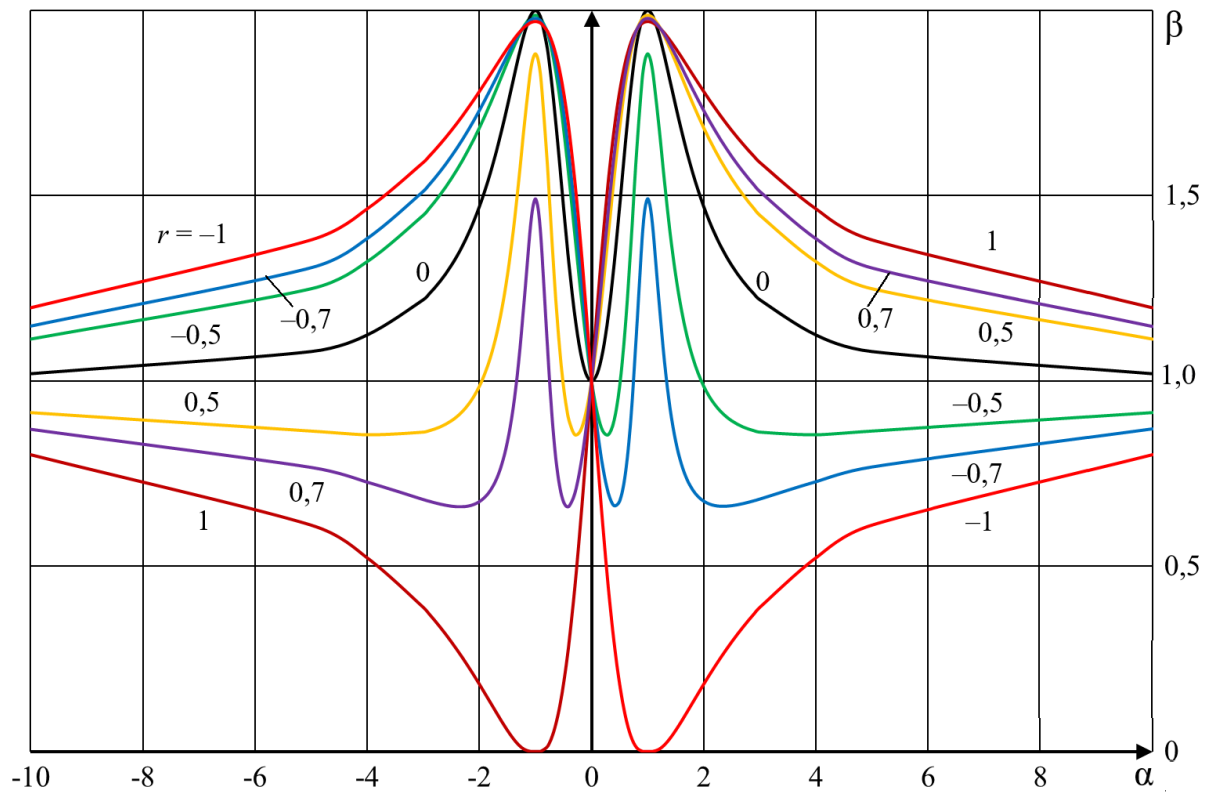


Fig. 2. Dependences of the coefficient β in expression (16) on α when changing the correlation coefficient in the range $-1 \leq r \leq 1$ for $n = 5$

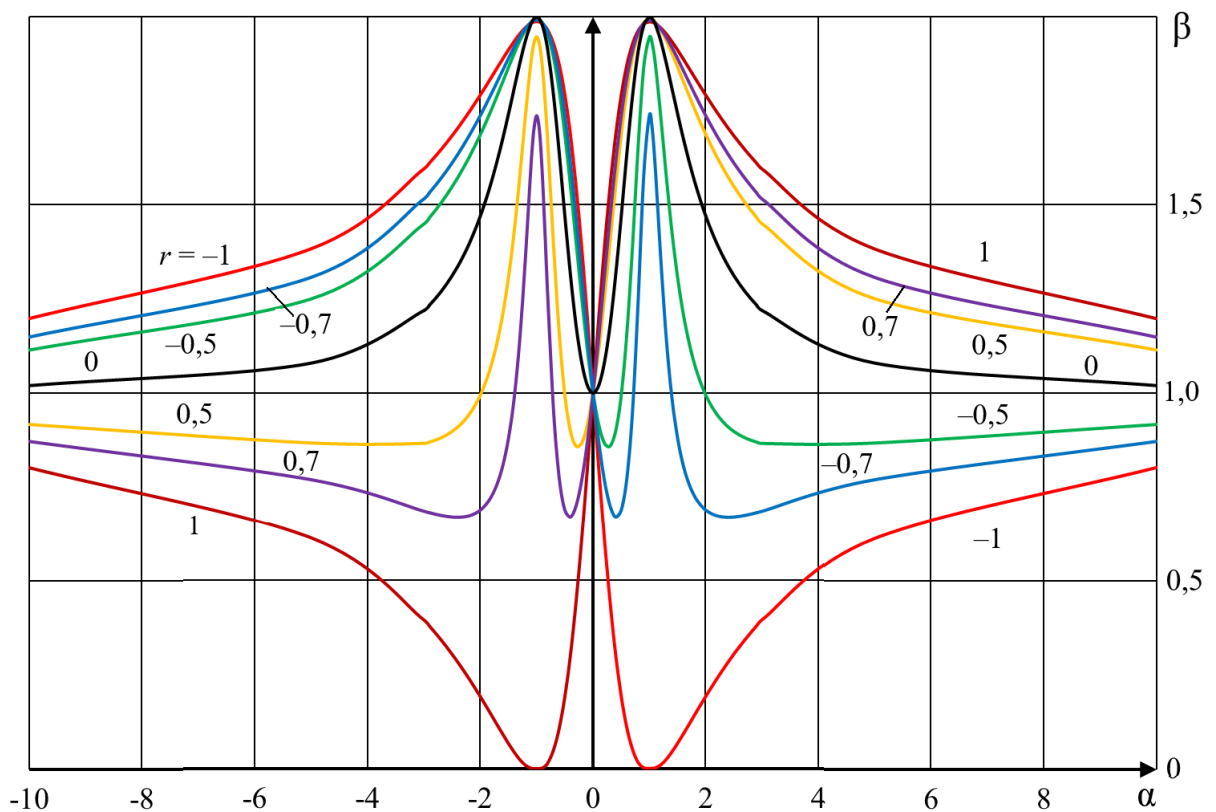


Fig. 3. Dependences of the coefficient β in expression (16) on α when changing the correlation coefficient in the range $-1 \leq r \leq 1$ for $n = 10$

of freedom $\nu = n - 1$. In this case, expression (4) for the combined standard uncertainty, given in the GUM, can be rewritten as follows:

$$u(y) = \sqrt{\sum_{i=1}^N u_{Bi}^2(y) + \sum_{\substack{i=1, \\ j=l,m}}^N u_{Ai}^2(y) + [u_{Ai}^2(y) + 2r_{i,m}u_{Ai}(y)u_{Am}(y) + u_{Am}^2(y)]}. \quad (17)$$

Then, the modified W-S equation, in the presence of two correlated contributions of Type A uncertainty, $u_{Ai}(y)$, $u_{Am}(y)$, will have the form:

$$v_{eff}^* = \frac{u^4(y)}{\sum_{i=1}^N \frac{u_{Bi}^4(y)}{\nu_{Bi}} + \sum_{\substack{i=1, \\ j=l,m}}^N \frac{u_{Ai}^4(y)}{n_i - 1} + \frac{[u_{Ai}^2(y) + 2r_{i,m}u_{Ai}(y)u_{Am}(y) + u_{Am}^2(y)]^2}{n - 1}}. \quad (18)$$

Thus, for the selected test model (1), these effective degrees of freedom will be equal to:

$$v_{eff}^* = \frac{[u_{Ai}^2(y) + 2r_{i,m}u_{Ai}(y)u_{Am}(y) + u_{Am}^2(y)]^2}{[u_{Ai}^2(y) + 2r_{i,m}u_{Ai}(y)u_{Am}(y) + u_{Am}^2(y)]^2 / (n - 1)} = n - 1, \quad (19)$$

which coincides with Result 1 of [4].

Obviously, in the presence of three correlated Type A uncertainty contributions $u_{Ai}(y)$, $u_{Am}(y)$, and $u_{Ak}(y)$, the modified W-S equation will have the form:

$$v_{eff}^* = \frac{u^4(y)}{\sum_{i=1}^N \frac{u_{Bi}^4(y)}{\nu_{Bi}} + \sum_{\substack{i=1, \\ j=l,m,k}}^N \frac{u_{Ai}^4(y)}{n_i - 1} + \frac{u_{Ar}^4(y)}{n - 1}}, \quad (20)$$

where

$$u_{Ar}^4(y) = \frac{1}{n - 1} [u_{Ai}^2(y) + u_{Am}^2(y) + u_{Ak}^2(y) + 2r_{i,m}u_{Ai}(y)u_{Am}(y) + 2r_{i,k}u_{Ai}(y)u_{Ak}(y) + 2r_{k,m}u_{Am}(y)u_{Ak}(y)]^2. \quad (21)$$

If all N Type A uncertainty contributions are correlated, then the modified W-S equation can be written in the general form as:

$$v_{eff}^* = \frac{u^4(y)}{\sum_{i=1}^N \frac{u_{Bi}^4(y)}{\nu_{Bi}} + \left[\sum_{i=1}^N u_{Ai}^2(y) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^N r_{i,j}u_{Aj}(y)u_{Ai}(y) \right]^2}. \quad (22)$$

The uncertainty budget for implementing this method for two correlated quantities X_l and X_m is presented in Table 1.

It should be noted that in the absence of correlation between the results of multiple observations of the input quantities, to calculate the effective degrees of freedom, it is necessary to use the basic W-S equation (6) given in the GUM instead of substituting the zero value of the correlation coefficient into expressions (18), (19), (22).

Table 1
Uncertainty budget for implementing the method proposed by the Authors for two correlated quantities X_l and X_m

Input quantities	Estimations of input quantities	Standard uncertainties	Degrees of freedom	Sensitivity coefficients	Uncertainty contributions
X_1	x_1	u_{A1}	$\nu_{A1} = n_1 - 1$	c_1	$u_{A1}(y)$
		u_{B1}	ν_{B1}		$u_{B1}(y)$
X_2	x_2	u_{A2}	$\nu_{A2} = n_2 - 1$	c_2	$u_{A2}(y)$
		u_{B2}	ν_{B1}		$u_{B2}(y)$
....
X_l	x_l	u_{Al}	$\nu_{Al} = n - 1$	c_l	$u_{Al}(y)$
		u_{Bl}	ν_{Bl}		$u_{Bl}(y)$
....
X_m	x_m	u_{Am}	$\nu_{Am} = n - 1$	c_m	$u_{Am}(y)$
		u_{Bm}	ν_{Bm}		$u_{Bm}(y)$
....
X_N	x_N	u_{AN}	$\nu_{AN} = n_N - 1$	c_N	$u_{AN}(y)$
		u_{BN}	ν_{BN}		$u_{BN}(y)$
Measurand	Measurand estimation	Combined standard uncertainty	Effective degrees of freedom	Coverage factor	Expanded uncertainty
Y	(3)	(17)	(18)	$t_{p;v_{eff}^*}$	U

6. Reduction method

The reduction method [7] is used when there is an observed correlation between the results of multiple simultaneous observations of two or more input quantities. It involves reducing indirect measurements to direct ones by calculating the values of the measured quantity for each pair of correlated input quantities.

For example, if, out of N input quantities, the observation results of the l -th and m -th quantities are obtained simultaneously, then the implementation scheme for the reduction method can be represented as shown in Table 2.

For function (1) of two correlated input quantities $Y = f(X_1, X_2)$ with equal degrees of freedom $\nu_1 = \nu_2 = \nu = n - 1$ and in the absence of type B un-

certainties, the effective degrees of freedom obtained by the reduction method will be:

$$\nu_r = \frac{u_r^4(y)}{u_r^4(y)/(n-1)} = n - 1. \quad (28)$$

It should be noted that with few observations of input quantities, an imaginary correlation between their results may arise even when there is no real reason for it. Applying the reduction method in this case will yield unreliable estimates of measurement uncertainty. Therefore, before applying the reduction method, the presence or absence of a correlation shall be verified using statistical criteria.

The uncertainty budget for implementing the reduction method for two correlated quantities X_l and X_m is presented in Table 3.

Table 2

Scheme of implementation of the reduction method

Input quantities								Measurand
X_1	X_2	...	X_l	...	X_m	...	X_N	$Y = f(X_1, X_2, \dots, X_l, \dots, X_m, \dots, X_N)$
x_1	x_2	...	x_{l1}	...	x_{m1}	...	x_N	$y_1 = f(x_1, x_2, \dots, x_{l1}, \dots, x_{m1}, \dots, x_N)$
			x_{l2}		x_{m2}			$y_2 = f(x_1, x_2, \dots, x_{l2}, \dots, x_{m2}, \dots, x_N)$
		
			x_{ln}		x_{mn}			$y_n = f(x_1, x_2, \dots, x_{ln}, \dots, x_{mn}, \dots, x_N)$
Measurand estimation								$y = \bar{y} = \frac{1}{n} \sum_{q=1}^n y_q$ (23)
Type A standard uncertainty of correlated quantities								$u_r(y) = \sqrt{\frac{1}{n(n-1)} \sum_{q=1}^n (y_q - \bar{y})^2}$ (24)
Combined standard uncertainty								$u(y) = \sqrt{\sum_{i=1}^N u_{Bi}^2(y) + \sum_{\substack{i=1, \\ i \neq l, m}}^N u_{Ai}^2(y) + u_r^2(y)}$ (25)
Expanded uncertainty of measurand								$U(y) = t_{p, \nu_{eff}} u(y)$ (26)
Effective degrees of freedom								$\nu_{eff} = \frac{u^4(y)}{\sum_{i=1}^N \frac{u_{Bi}^4(y)}{\nu_{Bi}} + \sum_{i=1, i \neq l, m}^N \frac{u_{Ai}^4(y)}{n_i - 1} + \frac{u_r^4(\bar{y})}{n - 1}}$ (27)

Table 3

Uncertainty budget for the implementation of the reduction method for two correlated quantities X_l and X_m

Input quantities	Estimations of input quantities	Standard uncertainties	Degrees of freedom	Sensitivity coefficients	Uncertainty contributions
X_1	x_1	u_{A1}	$\nu_{A1} = n_1 - 1$	c_1	$u_{A1}(y)$
		u_{B1}	ν_{B1}		$u_{B1}(y)$
X_2	x_2	u_{A2}	$\nu_{A2} = n_2 - 1$	c_2	$u_{A2}(y)$
		u_{B2}	ν_{B1}		$u_{B2}(y)$
....
X_l	x_l	u_{Bl}	ν_{Bl}	c_l	$u_{Bl}(y)$
....
X_m	x_m	u_{Bm}	ν_{Bm}	c_m	$u_{Bm}(y)$
....
X_N	x_N	u_{AN}	$\nu_{AN} = n_N - 1$	c_N	$u_{AN}(y)$
		u_{BN}	ν_{BN}		$u_{BN}(y)$
Measurand	Measurand estimation	Combined standard uncertainty	Effective degrees of freedom	Coverage factor	Expanded uncertainty
Y	(23)	(25)	(27)	$t_{p, \nu_{eff}}$	(26)

7. Example: evaluating the uncertainty of electrical resistance measurements using the voltmeter-ammeter method

Let us consider a method for evaluating the uncertainty of electrical resistance measurements by simultaneously measuring electrical voltage and current for five times (Table 4) using the GUM procedure [1], the method proposed by the Authors [2], and the reduction method [7].

Table 4

Results of simultaneous measurements of electrical voltage V and current I

N_0	V , mV	I , mA	R , Ohm
1	0.9129	0.9075	1.005928
2	0.9787	1.0449	0.936663
3	0.9166	0.9757	0.939338
4	0.9543	0.9902	0.963666
5	0.9496	0.9382	1.012065
mean	0.94239	0.97132	0.9715
u_A	0.01235	0.02341	0.01603

The correlation coefficient between the results of repeated observations V and I was 0.77.

The uncertainty budgets for resistor resistance measurements, processed using the proposed method

and the reduction method, are presented in Tables 5 and 6, respectively.

Comparative results of the measurement uncertainty evaluation obtained by all three methods are presented in Table 7.

From Table 7, it can be seen that the results obtained by the reduction method are close to the results obtained by the method proposed by the Authors and are almost twice as different from the results obtained by the W-S equation given in GUM.

Conclusions

1. The W-S equation presented in the GUM does not account for the correlation when calculating effective degrees of freedom, which leads to incorrect estimates of the expanded uncertainty of correlated measurements.

2. A test of approaches to estimating effective degrees of freedom described in [4, 5], in the presence of observed correlation between the measurement results of input quantities, revealed a discrepancy between the obtained results and those obtained using the reduction method [7], which is applied in the presence of observed correlation between the results of multiple simultaneous observations of two or more input quantities.

Table 5

Uncertainty budget for resistor resistance measurements when processed using the method proposed by the Authors

Input quantities	Estimations of input quantities	Standard uncertainties	Degrees of freedom	Sensitivity coefficients	Uncertainty contributions Ohm
I	0.971 mA	0.0234 mA	4	-0.999	-0.0234
		0.01 mA	∞		-0.00999
V	0.942 mV	0.01235 mV	4	1.030	0.0127
		0.01 mV	∞		0.0103
Measurand	Measurand estimation	Combined standard uncertainty	Effective degrees of freedom	Coverage factor	Expanded uncertainty, Ohm
R	0.970 Ohm	0.0214 Ohm	13	2.21	0.0474

Table 6

Uncertainty budget for resistor resistance measurements when processed using the reduction method

Input quantities	Estimations of input quantities	Standard uncertainties	Degrees of freedom	Sensitivity coefficients	Uncertainty contributions Ohm
I	0.971 mA	0.01 mA	∞	-0.999	-0.00999
V	0.942 mV	0.01 mV	∞	1.030	0.0103
Measurand	Measurand estimation	Combined standard uncertainty	Effective degrees of freedom	Coverage factor	Expanded uncertainty, Ohm
R	0.9715 Ohm	0.0215 Ohm	13	2.21	0.048 Ohm

Table 7

Results obtained using the GUM procedure, the method proposed by the Authors, and the reduction method

Method	Measurand value, Ohm	Measurand standard uncertainty, Ohm	Effective degrees of freedom	Coverage factor	Expanded uncertainty, Ohm
Welch-Satterthwaite	0.970	0.0214	2.6	4.53	0.097
Zakharov-Botsiura	0.970	0.0214	13	2.21	0.047
Reduction	0.9715	0.0215	13	2.21	0.048

3. To correctly calculate effective degrees of freedom in the presence of observed correlation between the results of multiple simultaneous observations of two or more input quantities, the Authors proposed modified expressions for the W-S equation.

4. The example studied suggested that the results obtained by the method proposed by the Authors and the reduction method are close to each other and differ significantly from the results obtained using the W-S equation presented in the GUM.

Аналіз підходів до обчислення ефективних степенів свободи за наявності кореляції між результатами вимірювання вхідних величин

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Анотація

Проаналізовано відомі підходи до визначення ефективного числа степенів свободи при оцінюванні розширеної невизначеності корельованих вимірювань. Як тестову модель обрано найпростішу математичну модель із двома вхідними величинами, результатам n одночасних, розподілених за нормальним законом спостережень котрих приписані лише невизначеності типу А. Вимірюваній величині для цієї моделі відповідає двомірний закон розподілу Стюдента з числом степенів свободи $(n-1)$.

Описано підхід GUM для оцінювання невизначеності корельованих вимірювань. Аналіз наведеного в GUM рівняння Велча-Саттерсвейта показав, що для прийнятої тестової моделі розраховане за цією формулою число степенів свободи буде змінюватися від нуля до $8(n-1)$ за зміни значення коефіцієнта кореляції в границях ± 1 та співвідношення внесків невизначеності вхідних величин у границях $\pm\infty$. У двох точках цієї залежності ефективне число степенів свободи набуває нульового значення, що призводить до неприпустимого нескінченного значення розширеної невизначеності.

У статтях авторів R. Willink та H. Castrup були отримані вирази для розширення формули Велча-Саттерсвейта на корельовані вимірювання. Аналіз цих виразів за допомогою вибраної тестової моделі показав суттєву відмінність отриманих за ними ефективних чисел степенів від $(n-1)$ за зміни значення коефіцієнта кореляції та співвідношення внесків невизначеностей вхідних величин, що ставить під сумнів справедливості цих виразів.

Проаналізовано запропоновану авторами формулу для розрахунку ефективного числа степенів свободи з урахуванням спостережуваної кореляції. Для прийнятої тестової моделі вона приводить до числа степенів свободи $(n-1)$.

Аналогічний результат отримано для методу редукції, розробленого спеціально для випадку корельованих вимірювань.

Розглянутий у статті приклад оцінювання розширеної невизначеності при вимірюванні опору резистора методом вольтметра-амперметра показав співпадіння результатів, отриманих запропонованим авторами методом та методом редукції.

Ключові слова: невизначеність вимірювань; кореляція; ефективне число степенів свободи; рівняння Велча-Саттерсвейта; метод редукції.

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